Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	12	12	100	92	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.
2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.
3. You must be careful to confine your answer to the specific question asked and to follow any instructions that are specified to a particular question.
4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.
5. It is recommended that you do not use pencil, except in diagrams.
6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.
7. The Formula sheet is not to be handed in with your Question/Answer booklet.

See next page

Section One: Calculator-free **35% (50 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

C_1 and C_2 are chords in the same circle. Let P be the statement 'chords C_1 and C_2 have equal lengths' and Q be the statement 'chords C_1 and C_2 subtend equal angles at the centre'.

(a) Write, in words, the negation of P. (1 mark)

Solution
Chords C_1 and C_2 do not have equal lengths.
Specific behaviours
✓ correct statement, in words

(b) Write, in words, the meaning of $P \Rightarrow Q$. (1 mark)

Solution
If chords C_1 and C_2 have equal lengths, then chords C_1 and C_2 subtend equal angles at the centre.
Specific behaviours
✓ correct statement, in words

(c) Write, symbolically, the converse of $P \Rightarrow Q$. (1 mark)

Solution
$Q \Rightarrow P$
Specific behaviours
✓ correct statement, in symbols

(d) Write, in any form, the inverse of $P \Rightarrow Q$. (1 mark)

Solution
$\neg P \Rightarrow \neg Q$; or 'If not P then not Q'; or 'If chords C_1 and C_2 do not have equal lengths, then chords C_1 and C_2 do not subtend equal angles at the centre'.
Specific behaviours
✓ correct statement

(e) Write, in any form, the contrapositive of $P \Rightarrow Q$ and state, with justification, whether the contrapositive is true. (2 marks)

Solution
$\neg Q \Rightarrow \neg P$; or 'If not Q then not P'; or 'If chords C_1 and C_2 do not subtend equal angles at the centre, then chords C_1 and C_2 do not have equal lengths'. The contrapositive statement is true.
Specific behaviours
✓ correct statement ✓ states true

See next page

Question 2 **(5 marks)**

Use the inclusion-exclusion principle to determine how many integers between 1 and 131 inclusive are divisible by 2, 3 or 8.

Solution

Divisible by 2; or by 3; or by 8:

$$\lfloor 131 \div 2 \rfloor = 65$$
$$\lfloor 131 \div 3 \rfloor = 43$$
$$\lfloor 131 \div 8 \rfloor = 16$$

Divisible by 2 and 3 (6); or by 2 and 8 (8); or by 3 and 8 (24):

$$\lfloor 131 \div 6 \rfloor = 21$$
$$\lfloor 131 \div 8 \rfloor = 16$$
$$\lfloor 131 \div 24 \rfloor = 5$$

Divisible by 2 and 3 and 8 (24):

$$\lfloor 131 \div 24 \rfloor = 5$$

Using the inclusion-exclusion principle:

$$\begin{aligned} n &= 65 + 43 + 16 - 21 - 16 - 5 + 5 \\ &= 65 + 43 - 21 \\ &= 87 \end{aligned}$$

Specific behaviours

✓ clearly indicates methodical approach
✓ at least two correct cases for divisible singly
✓ at least two correct cases for divisible by pairs
✓ correct number for divisible by all three
✓ obtains correct number of integers, no errors
(NB No marks for correct answer if no evidence of use of inclusion-exclusion principle)

Question 3 **(7 marks)**

Two vectors are $\mathbf{a} = 6\mathbf{i} - 8\mathbf{j}$ and $\mathbf{b} = \mathbf{i} - 3\mathbf{j}$.

(a) Determine the vector projection of $\mathbf{a}$ on $\mathbf{b}$. (3 marks)

Solution

$$\frac{\begin{pmatrix} 6 \\ -8 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix}}{\begin{pmatrix} 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix}} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \frac{30}{10} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$
$$= 3 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ -9 \end{pmatrix}$$

Specific behaviours

✓ indicates correct method
✓ correct scalar products
✓ correct vector

(b) Vector $\mathbf{c} = x\mathbf{i} + y\mathbf{j}$ has twice the magnitude of $\mathbf{a}$ and is perpendicular to $\mathbf{b}$. Determine the values of the constants x and y. (4 marks)

Solution

Require $\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \end{pmatrix} = 0$ and $\left|\begin{pmatrix} x \\ y \end{pmatrix}\right| = 2\left|\begin{pmatrix} 6 \\ -8 \end{pmatrix}\right|$.

Hence $x - 3y = 0$ and $x^2 + y^2 = 4(6^2 + (-8)^2) = 400$.

$$\begin{aligned} (3y)^2 + y^2 &= 400 \\ 10y^2 &= 400 \\ y &= \pm 2\sqrt{10}, \qquad x = 3y \end{aligned}$$

Hence $x = 6\sqrt{10}, y = 2\sqrt{10}$ or $x = -6\sqrt{10}, y = -2\sqrt{10}$.

Specific behaviours

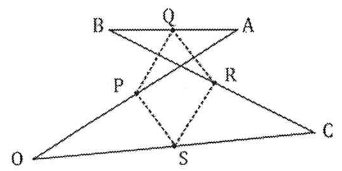
✓ equation using perpendicular
✓ equation using magnitude
✓ solves for one constant
✓ states both solution sets

Question 4 **(8 marks)**

Crossed quadrilateral $OABC$ is shown in the diagram.

The midpoints of sides OA, AB, BC and CO are P, Q, R and S respectively.

Let $\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.



(a) Draw quadrilateral $PQRS$ on the diagram. (1 mark)

(b) Determine expressions for $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}$ and $\overrightarrow{OS}$ in terms of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. (3 marks)

Solution (a) (b)
(a) see diagram for $PQRS$.
(b)
$\overrightarrow{OP} = \frac{1}{2}\mathbf{a}, \quad \overrightarrow{OS} = \frac{1}{2}\mathbf{c}$
$\overrightarrow{OQ} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$
$\overrightarrow{OR} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CB} = \mathbf{c} + \frac{1}{2}(\mathbf{b} - \mathbf{c}) = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c}$
Specific behaviours
✓ (a) correct midpoints, joined to form $PQRS$
✓ expressions for $\overrightarrow{OP}, \overrightarrow{OS}$
✓ expression for $\overrightarrow{OQ}$
✓ expression for $\overrightarrow{OR}$

(c) Prove that the midpoints of the sides of a crossed quadrilateral join to form a parallelogram. (4 marks)

Solution
$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a} = \frac{1}{2}\mathbf{b}$
$\overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{c} = \frac{1}{2}\mathbf{b}$
Since $\overrightarrow{PQ} = \overrightarrow{SR}$ then $PQRS$ has a pair of congruent and parallel sides and is a parallelogram. Hence, the midpoints of the sides of a crossed quadrilateral join to form a parallelogram.
(NB can use other pair of sides, different reasoning, etc)
Specific behaviours
✓ expression for $\overrightarrow{PQ}$
✓ expression for $\overrightarrow{SR}$
✓ correctly reasons that $PQRS$ is a parallelogram
✓ completes proof

<u>Alternate</u>

$\overrightarrow{PS} = \overrightarrow{PO} + \overrightarrow{OS}$

$= -\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$

$\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$

$= -\frac{1}{2}\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{c}$

$= -\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$

$\overrightarrow{PS} \parallel \overrightarrow{QR}$ and $\overrightarrow{PS} = \overrightarrow{QR}$

$\therefore$ PQRS is parallelogram.

Question 5 **(8 marks)**

Let $\mathbf{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$.

(a) Determine

(i) $4\mathbf{a} + 2\mathbf{c}$. (2 marks)

Solution
$4\mathbf{a} + 2\mathbf{c} = 4\begin{pmatrix} 3 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 12 \\ 20 \end{pmatrix} + \begin{pmatrix} 8 \\ -10 \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \end{pmatrix}$
Specific behaviours
✓ correct multiples
✓ correct vector

(ii) $\lvert\mathbf{c} - \mathbf{b}\rvert$. (2 marks)

Solution
$\mathbf{c} - \mathbf{b} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$
$\lvert\mathbf{c} - \mathbf{b}\rvert = \sqrt{1+9} = \sqrt{10}$
Specific behaviours
✓ difference of vectors
✓ correct magnitude

(b) Given that $\mathbf{a} = \lambda\mathbf{b} + \mu\mathbf{c}$, determine the value of the constant λ and the value of the constant μ. (4 marks)

Solution
$\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \lambda\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} 4 \\ -5 \end{pmatrix}$
Equating $\mathbf{i}, \mathbf{j}$ coefficients $3\lambda + 4\mu = 3 \rightarrow 6\lambda + 8\mu = 6$; $-2\lambda - 5\mu = 5 \rightarrow -6\lambda - 15\mu = 15$
Add equations $-7\mu = 21 \rightarrow \mu = -3$
Substitute $3\lambda + 4(-3) = 3 \rightarrow \lambda = 5$
Hence $\lambda = 5, \quad \mu = -3$
Specific behaviours
✓ equates i-coefficients
✓ equates i-coefficients
✓ solves for one constant
✓ both correct constants

Question 6 **(8 marks)**

(a) ABC is an isosceles triangle in which $BA = BC$. If M is the midpoint of AC, use a vector method to show that $\overrightarrow{BA} + \overrightarrow{BC} = 2\overrightarrow{BM}$. (3 marks)

Solution

$$\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM} \quad (1)$$
$$\overrightarrow{BM} = \overrightarrow{BC} + \overrightarrow{CM} = \overrightarrow{BC} - \overrightarrow{MC} = \overrightarrow{BC} - \overrightarrow{AM} \quad (2)$$

(Since $\overrightarrow{AM} = \overrightarrow{MC}$ as M is midpoint of AC)

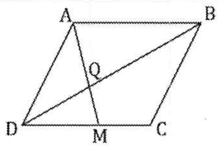
Adding $(1) + (2)$
$$\overrightarrow{BM} + \overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM} + \overrightarrow{BC} - \overrightarrow{AM}$$
$$2\overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{BC}$$

Specific behaviours

✓ two equations for $\overrightarrow{BM}$
✓ uses midpoint to eliminate $\overrightarrow{AM}$ or $\overrightarrow{CM}$
✓ adds equations and simplifies

(b) $ABCD$ is a parallelogram and M is the midpoint of DC. Diagonal DB intersects AM at Q so that $\overrightarrow{AQ} = h\overrightarrow{AM}$ and $\overrightarrow{DQ} = k\overrightarrow{DB}$. Use a vector method to determine the value of the constant h and the value of the constant k. (5 marks)

Solution



$$\overrightarrow{DQ} = \overrightarrow{DA} + \overrightarrow{AQ}$$
$$= \overrightarrow{DA} + h\overrightarrow{AM}$$
$$= \overrightarrow{DA} + h\left(\frac{1}{2}\overrightarrow{DC} - \overrightarrow{DA}\right)$$
$$= (1-h)\overrightarrow{DA} + \frac{h}{2}\overrightarrow{DC} \quad (1)$$

But also
$$\overrightarrow{DQ} = k\overrightarrow{DB}$$
$$= k(\overrightarrow{DA} + \overrightarrow{DC})$$
$$= k\overrightarrow{DA} + k\overrightarrow{DC} \quad (2)$$

Equating (1) and (2)
$$k\overrightarrow{DA} + k\overrightarrow{DC} = (1-h)\overrightarrow{DA} + \frac{h}{2}\overrightarrow{DC}$$

Equating vector coefficients
$$k = 1 - h \text{ and } k = \frac{h}{2} \Rightarrow \frac{h}{2} = 1 - h \Rightarrow h = \frac{2}{3}, \qquad k = \frac{1}{3}$$

Specific behaviours

✓ sketch diagram
✓ expresses $\overrightarrow{DQ}$ in terms of $\overrightarrow{DA}$ and $\overrightarrow{AM}$
✓ obtains equation (1)
✓ obtains equation (2)
✓ equates coefficients to obtain value of each constant
(*Many variations exist using different sides or* $\overrightarrow{DA} = \mathbf{a}$, *etc*)

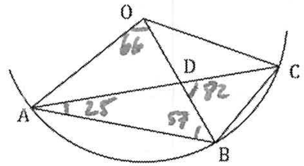
See next page

Question 7 **(8 marks)**

(a) Points A, B and C lie on an arc of a circle with centre O as shown at right.

Chord AC intersects OB at point D.

The diagram is not drawn to scale.



When $\angle ABC = 122°$ and $\angle BCA = 33°$, determine the size of $\angle BDC$. (4 marks)

Solution

$$\angle AOB = 2\angle BCA = 2 \times 33° = 66°$$
$$\angle BAC = 180° - 122° - 33° = 25°$$
$$\angle OBA = \frac{1}{2}(180° - 66°) = 57°$$
$$\angle BDC = 25° + 57° = 82°$$

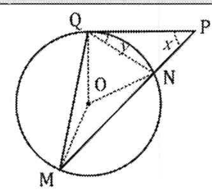
Specific behaviours

✓ obtains $\angle AOB$
✓ obtains $\angle BAC$
✓ obtains $\angle OBA$
✓ correct $\angle BDC$

(b) A secant cuts a circle with centre O at points M and N. Secant MN is extended beyond N to point P, where it meets a line that is a tangent to the circle at point Q. Prove that the size of $\angle NPQ$ is equal to one half the difference of the sizes of $\angle MOQ$ and $\angle NOQ$. (4 marks)

Solution

Let $\angle NPQ = x$ and $\angle PQN = y$. Then



$$\angle MNQ = x + y \quad (1)$$
$$\angle MOQ = 2x + 2y \quad (2)$$
$$\angle NMQ = y \quad (3)$$
$$\angle NOQ = 2y \quad (2)$$

Hence
$$\frac{1}{2}(\angle MOQ - \angle NOQ) = \frac{1}{2}(2x + 2y - 2y)$$
$$= x$$
$$= \angle NPQ$$

Reasoning:
(1) sum of two interior angles is equal to the opposite exterior angle
(2) angle at centre property
(3) angle in alternate segments

Specific behaviours

✓ reasonable diagram (variations exist - secant between O and Q, etc.)
✓ derives expression for $\angle MOQ$
✓ derives expression for $\angle NOQ$
✓ completes proof, with reasonable explanation throughout

End of questions

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Section Two: Calculator-assumed **65% (92 Marks)**

This section has **twelve** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

Question 8 **(6 marks)**

A classroom has a barrel containing a large number of blue crayons and yellow crayons.

(a) 14 crayons are randomly drawn from the barrel and placed, in the order, on a table. In terms of their colours, how many different orders are possible? (1 mark)

Solution
$2^{14} = 16\,384$
Specific behaviours
✓ correct number

(b) An empty box is to be filled with some of the crayons. What is the smallest number of crayons that should be placed in the box to guarantee that it contains at least 7 blue or at least 9 yellow crayons? Justify your answer. (2 marks)

Solution
Take two pigeonholes and fill one with 6 blue and the other with 8 yellow. When one more crayon is chosen and placed in its pigeonhole, then one of the conditions will be met and so $6+8+1=15$ is the smallest number of crayons that should be placed in the box.
Specific behaviours
✓ correct smallest number ✓ justification

(c) Each of the 21 students in the classroom choose 5 crayons from the barrel. Use the pigeonhole principle to prove that at least 4 of the students will choose the same colour combination of crayons. (3 marks)

Solution
All possible combinations of crayons represent the pigeonholes, so that there are 6 pigeonholes $(0, 1, 2, 3, 4$ or 5 blues in selection). The pigeons are the selections made by each student, so that there are 21 pigeons. Using the pigeonhole principle, there will be at least $\lceil 21 \div 6 \rceil = 4$ students who choose the same colour combination.
Specific behaviours
✓ relates pigeonholes and pigeons to context ✓ correctly identifies number of pigeonholes ✓ completes proof using pigeonhole principle

Question 9 **(6 marks)**

Four figure numbers are to be formed from the digits $1, 2, 3, 4, 5$.

(a) How many different four figure numbers can be formed

(i) if repetition of digits is allowed? (1 mark)

Solution
$5^4 = 625$
Specific behaviours
✓ correct number

(ii) without repetition of digits? (1 mark)

Solution
$5 \times 4 \times 3 \times 2 = 120$
Specific behaviours
✓ correct number

(b) How many of the numbers without repetition are greater than 4312? (3 marks)

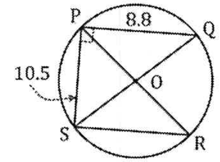
Solution
Start with 4315: 1 Start with 432 or 435: $2 \times 2! = 4$ Start with 45: $1 \times 3! = 6$ Start with 5: $1 \times 4! = 24$ Total numbers: $1 + 4 + 6 + 24 = 35$
Specific behaviours
✓ splits into cases ✓ correct count for at least two cases ✓ correct number

(c) How many of the numbers without repetition are less than 4312? (1 mark)

Solution
$120 - 35 - 1 = 84$
Specific behaviours
✓ correct number

Question 10 **(9 marks)**

(a) Points P, Q, R and S lie in order on the circumference of the circle with centre O so that $PQ = 8.8$ cm, $PS = 10.5$ cm, and PR and QS are diameters. Determine, with brief reasons and to the nearest degree, the sizes of $\angle PQS$, $\angle PRS$, $\angle POS$ and $\angle RPS$. (5 marks)

Solution	
$\angle PQS = \tan^{-1}\left(\frac{10.5}{8.8}\right) = 50°$ $\angle PRS = \angle PQS = 50°$ (angle on same arc) $\angle POS = 2\angle PQS = 100°$ (angle at centre) $\angle RPS = 90° - 50° = 40°$ (angle in semicircle)	
Specific behaviours	
✓ labelled diagram showing diameters, chords, lengths ✓✓✓✓ calculates each angle, with reasoning	

(b) Points A, B and C lie on the circumference of a circle of radius 26 cm, so that $BC = 28$ cm and $AC = 45$ cm. Prove by contradiction that the midpoint of chord AB is not the centre of the circle. (4 marks)

Solution
Assume that the midpoint of AB is the centre of the circle. Hence AB is a diameter of the circle and the angle in a semicircle theorem implies that ΔABC must be right angled at C. Using Pythagoras' theorem, the length of diameter AB is given by $AB = \sqrt{AC^2 + BC^2}$ $= \sqrt{45^2 + 28^2}$ $= 53$ cm Hence the radius of the circle is $53 \div 2 = 26.5$ cm. This result contradicts the fact that the radius of the circle is 26 cm and so our assumption is wrong and thus the midpoint of chord AB is not the centre of the circle.
Specific behaviours
✓ states assumption ✓ uses assumption to imply that ΔABC is right angled ✓ calculates diameter of circle ✓ uses contradiction to complete proof

Question 11 **(8 marks)**

Relative to boat O at anchor in a lake, four buoys A, B, C and D have the following position vectors (with distances in metres):

$\overrightarrow{OA} = (210, -935)$, $\overrightarrow{OB} = (90, -200)$, $\overrightarrow{OC} = (330, 360)$, $\overrightarrow{OD} = (390, -515)$.

(a) Prove that the quadrilateral with vertices $ABCD$ is a trapezium, but not a parallelogram. (5 marks)

Solution
Displacement vectors for all four sides are $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (-120, 735)$ $\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = (-60, 875)$ $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (240, 560)$ $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = (180, 420)$ Using $\mathbf{i}$-coefficients, $\frac{240}{180}\overrightarrow{AD} = (240, 560) = \overrightarrow{BC}$ and hence $\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$. Also, $\frac{120}{60}\overrightarrow{DC} = (-120, 1750) \neq \overrightarrow{AB}$ and hence $\overrightarrow{DC}$ is not parallel to $\overrightarrow{AB}$. Hence $ABCD$ has just one pair of parallel sides and thus is a trapezium but not a parallelogram.
Specific behaviours
✓ calculates correct displacement vectors for at least one side ✓ calculates correct displacement vectors for all sides ✓ clearly shows $\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$. ✓ clearly shows $\overrightarrow{DC}$ is not parallel to $\overrightarrow{AB}$ ✓ uses results to justify $ABCD$ is a trapezium but not a parallelogram

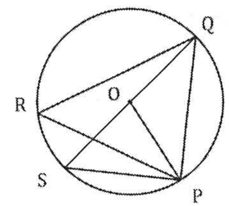
(b) Boat X motors directly from D to B with a constant velocity in 2 minutes and 30 seconds. Determine the velocity in component form, and hence the speed, of boat X. (3 marks)

Solution
Displacement vector $\overrightarrow{DB} = \overrightarrow{OB} - \overrightarrow{OD} = (-300, 315)$ m. Hence velocity vector $\mathbf{v}_{DB} = \overrightarrow{DB} \div 150 = (-2, 2.1)$ m/s. Hence speed $= \lvert\mathbf{v}_{DB}\rvert = 2.9$ m/s.
Specific behaviours
✓ displacement vector ✓ velocity vector ✓ speed, with units.

174 m/min

Question 12 **(8 marks)**

(a) In the diagram (not to scale), points P, Q, R and S lie on a circle with centre O and diameter QS.



If $\angle QRP = 59°$, determine the size of

(i) $\angle QOP$. (1 mark)

(ii) $\angle QSP$. (1 mark)

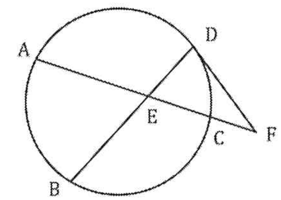
(iii) $\angle SQP$. (1 mark)

(iv) $\angle POS$. (1 mark)

Solution
$\angle QOP = 2\angle QRP = 118°$ $\angle QSP = \angle QRP = 59°$ $\angle SQP = 90° - \angle QSP = 31°$ $\angle POS = 2\angle SQP = 62°$
Specific behaviours
✓✓✓✓ each correct angle

(b) In the diagram (not to scale), FD is a tangent to the circle at D. Secant AF cuts chord BD at E, and the circle at C.

$FC = 4$ cm, $CA = 21$ cm, $DF = EF$, and BE is 2 cm longer than DF.



Determine the length of DE. (4 marks)

Solution
Secant-tangent property: $CF \times AF = DF^2$ $(21 + 4) \times 4 = DF^2$ $DF = 10$ cm Intersecting chords property: $BE \times DE = AE \times CE$ $(10 + 2) \times DE = (21 - (10 - 4))(10 - 4)$ $10DE = 15 \times 6$ $DE = 7.5$ cm
Specific behaviours
✓ correct use of secant-tangent property ✓ length DF ✓ correct use of intersecting chords property ✓ length DE

Question 13 **(7 marks)**

(a) Consider the letters in the word DEMATERIALISE. Determine the number of different

(i) combinations of 3 letters chosen from the consonants in the word. (1 mark)

Solution
$\binom{6}{3} = 20$
Specific behaviours
✓ correct number

(ii) permutations of all the letters in the word. (2 marks)

Solution
$n = \dfrac{13!}{3!\,2!\,2!} = 259\,459\,200$
Specific behaviours
✓ reasonable attempt to deal with repeated letters ✓ correct number

(b) Four-digit pin codes such as 3812 are made by randomly choosing four different digits from those in the number 12 345 678. Determine the fraction of all such possible pin codes that start with 12 or end in 8. (4 marks)

Solution
If T are codes that start with 12, and N are codes that end in 8, then $n(T) = 1 \times 1 \times 6 \times 5 = 30$ $n(N) = 7 \times 6 \times 5 \times 1 = 210$ $n(N \cap T) = 1 \times 1 \times 5 \times 1 = 5$ $n(N \cup T) = 30 + 210 - 5 = 235$ Total number of codes is ${}^8P_4 = 1680$. Hence fraction of all codes is $\dfrac{235}{1680} = \dfrac{47}{336}$
Specific behaviours
✓ calculates $n(T), n(N)$ ✓ calculates $n(N \cap T)$ ✓ uses inclusion-exclusion for $n(N \cup T)$ ✓ number of all possible codes and correct fraction

Question 14 **(8 marks)**

Two vectors are $\mathbf{a} = \begin{pmatrix} x \\ -5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ y \end{pmatrix}$, where x and y are constants.

(a) When $x = 11$ and $y = 7$, determine

(i) $\mathbf{a} \cdot \mathbf{b}$. (1 mark)

Solution
$\begin{pmatrix} 11 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 7 \end{pmatrix} = 31$
Specific behaviours
✓ correct value

(ii) a unit vector in the same direction as $\mathbf{a} - \mathbf{b}$, in exact form. (2 marks)

Solution
$\mathbf{u} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} 11 \\ -5 \end{pmatrix} - \begin{pmatrix} 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}, \quad \hat{\mathbf{u}} = \begin{pmatrix} 5/13 \\ -12/13 \end{pmatrix}$
Specific behaviours
✓ indicates $\mathbf{a} - \mathbf{b}$ ✓ correct unit vector

(iii) the angle between the directions of the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, rounded to one decimal place. (2 marks)

Solution
NB the direction of a vector and its unit vector are the same. Using CAS, $\theta = 73.843° \approx 73.8°$ to one dp.
Specific behaviours
✓ indicates method ✓ correct angle, as required

(b) Determine the value of x and the value of y when $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $x < y$, and $|\mathbf{a} + \mathbf{b}| = 12.2$. (3 marks)

Solution
$\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow 6x - 5y = 0$ $\mathbf{a} + \mathbf{b} = \begin{pmatrix} x+6 \\ y-5 \end{pmatrix}$ $\lvert\mathbf{a} + \mathbf{b}\rvert = 12.2 \Rightarrow (x+6)^2 + (y-5)^2 = 12.2^2$ Solving simultaneously for solution where $x < y$: $x = 6, \quad y = 7.2$
Specific behaviours
✓ obtains equation using scalar product ✓ obtains equation using sum of magnitudes ✓ states correct solution set

Question 15 **(8 marks)**

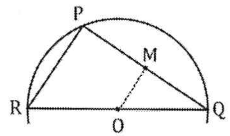
(a) The vertices of triangle PQR lie on a circle with centre O so that QR is a diameter. The midpoint of PQ is M. Prove that OM is parallel to RP. (4 marks)

Solution

Angle in semicircle: $\angle RPQ = 90°$.

Line from O bisecting chord is perpendicular to chord, so $\angle OMQ = 90°$.

Hence $\angle OMQ = \angle RPQ$ and since they are corresponding angles then OM is parallel to RP.



Specific behaviours

✓ diagram
✓ reasons for $\angle RPQ = 90°$
✓ reasons for $\angle OMQ = 90°$
✓ completes proof
NB A vector proof is also acceptable, where $\overrightarrow{RP} = 2\overrightarrow{OM}$ is shown

(b) The vertices of a kite lie on the circumference of a circle. Each longer side of the kite is four times the length of the adjacent shorter side. If the area of the kite is 64 cm^2, determine the radius of the circle. (4 marks)

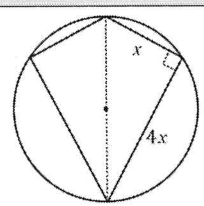
Solution

RHS of kite is right triangle (angle in semicircle).
Hence:

$$A = \frac{1}{2}(4x)(x) \times 2$$
$$4x^2 = 64$$
$$x = 4$$

Hence

$$(2r)^2 = 16^2 + 4^2$$
$$r = 2\sqrt{17} \approx 8.25 \text{ cm}$$

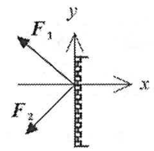


Specific behaviours

✓ reasonable diagram
✓ uses angles in semicircle property to form area equation
✓ solves equation
✓ correct radius

Question 16 **(9 marks)**

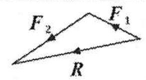
The diagram at right, not to scale, shows forces $\mathbf{F_1}$ and $\mathbf{F_2}$ acting in the same vertical plane on a small hook fixed to a vertical wall. $\mathbf{F_1}$ has magnitude 147 N and acts at an angle of elevation of 22° and $\mathbf{F_2}$ has magnitude 195 N and acts at an angle of depression of 42°.



The resultant of $\mathbf{F_1}$ and $\mathbf{F_2}$ is $\mathbf{R}$.

(a) Sketch a triangle to show the relationship between $\mathbf{F_1}$, $\mathbf{F_2}$ and $\mathbf{R}$. (1 mark)

Solution



Specific behaviours

✓ nose-to-tails force vectors and completes triangle with resultant

(b) Determine, with reasoning, the magnitude of $\mathbf{R}$ and the acute angle it makes with the wall. (5 marks)

Solution

Angle in triangle between forces is $180° - 22° - 42° = 116°$.

$$|\mathbf{R}| = 147^2 + 195^2 - 2(147)(195)\cos 116°$$
$$= 291.15 \approx 291 \text{ N}$$

Let θ be the angle between $\mathbf{F_1}$ and $\mathbf{R}$:

$$\frac{\sin\theta}{195} = \frac{\sin 116°}{291.15}$$
$$\theta = 37.0°$$

Hence acute angle with wall is $90° + 22° - 37° = 75°$.

Specific behaviours

✓ correct angle between forces (*shown here or in (a)*)
✓ expression using cosine rule with magnitude
✓ calculates magnitude
✓ expression using sine rule with angle
✓ calculates angle with horizontal

$\underset{\sim}{R} = \begin{pmatrix} 147 \\ \angle(158) \end{pmatrix} + \begin{pmatrix} 195 \\ \angle(-138) \end{pmatrix}$ ✓
$= \begin{pmatrix} -281.21 \\ -75.41 \end{pmatrix}$ ✓
toPol $\begin{pmatrix} -281.21 \\ -75.41 \end{pmatrix}$ ✓
$= \begin{bmatrix} 291.15 \\ \angle(-165) \end{bmatrix}$ ✓
angle ✓

(c) The wall exerts a force on the hook of equal magnitude to $\mathbf{R}$ but in the opposite direction. Express this force using unit vectors $\mathbf{i}$ and $\mathbf{j}$. (3 marks)

Solution

Angle $\mathbf{R}$ makes with x-axis is $-90° - 75° = -165°$.

$$\mathbf{R} = 291.15(\cos(-165°), \sin(-165°))$$
$$= -281.2\mathbf{i} - 75.4\mathbf{j}$$

Hence force exerted by wall is $281.2\mathbf{i} + 75.4\mathbf{j}$.

Specific behaviours

✓ indicates angle of resultant with x-axis, or similar
✓ converts into component form
✓ correctly reverses direction

Question 17 **(9 marks)**

A small boat has a cruising speed of 17 km/h in still water. The boat leaves point A at $8:30$ am and travels to point B, 7.7 km due east of A, where it turns and travels to point C, 3.6 km due north of B. The boat then returns to A. A current of 2.6 km/h runs in an easterly direction throughout the area.

(a) Determine the time taken to travel from point A to point B. (1 mark)

Solution
Time for leg AB: $t_{AB} = 7.7 \div (17 + 2.6) = 0.3929$ h.
Specific behaviours
✓ time for leg AB

= 23.574 mins

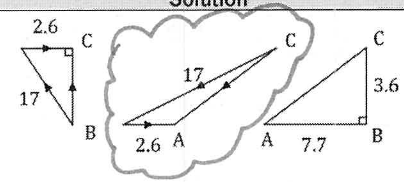
(b) Determine the time taken to travel from point B to point C. (2 marks)

Solution
Speed made good for leg BC: $s_{AB} = \sqrt{17^2 - 2.6^2} = 16.8$ km/h.   Time for leg BC: $t_{BC} = 3.6 \div 16.8 = 0.2143$ h.
Specific behaviours
✓ speed for leg BC ✓ time for leg BC

= 12.858 mins

(c) Determine the time taken for the return leg, from point C to point A. (5 marks)

Solution



$\angle A = \tan^{-1}\left(\frac{3.6}{7.7}\right) = 25.0576° \to 180° - \angle A$
$= 154.9424°$

Speed made good for leg CA:
$17^2 = s_{CA}^2 + 2.6^2 - 2(s_{CA})(2.6)\cos 154.9424°$
$s_{CA} = 14.61$

Distance CA: $CA = \sqrt{3.6^2 + 7.7^2} = 8.5$ km

Time for leg CA: $t_{CA} = 8.5 \div 14.61 = 0.5818$

Specific behaviours

✓ correct diagram for last leg (centre diagram)
✓ calculates angle in last leg triangle
✓ indicates correct use of trig to solve for s_{CA}
✓ speed for leg CA
✓ distance and time for leg CA

= 34.908 mins

(d) What time does the boat return to A? (1 mark)

Solution
Total time: $t = 0.3929 + 0.2143 + 0.5818 = 1.189$ $= 1{:}11'20''$ h.   Boat will return to A at $8:30 + 1:11 = 9:41$ am.
Specific behaviours
✓ correct return time

Question 18 **(6 marks)**

Consider the identity ${}^{n}C_r = {}^{(n+1)}C_r - {}^{n}C_{(r-1)}$.

(a) With $n = 4$ and $r = 3$, show that the left and right sides of the identity are equal. (1 mark)

Solution
$LHS = \binom{4}{3} = 4, \quad RHS = \binom{5}{3} - \binom{4}{2} = 10 - 6 = 4$ $\therefore LHS = 4 = RHS$
Specific behaviours
✓ correctly shows substitution and simplification

(b) State all necessary restrictions on n and r for the identity to exist and to be valid. (2 marks)

Solution
$n, r \in \mathbb{Z}, \quad n \geq r \geq 1$
Specific behaviours
✓ states at least one restriction (e.g., $r \geq 1$)   ✓ states all restrictions

(c) Prove the identity is always true, subject to all necessary restrictions. (3 marks)

Solution

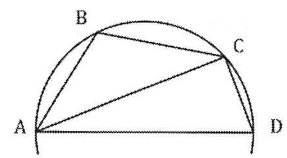
$$\begin{aligned}RHS &= {}^{n+1}C_r - {}^{n}C_{r-1}\\ &= \frac{(n+1)!}{r!\,(n+1-r)!} - \frac{n!}{(r-1)!\,(n-r+1)!}\\ &= \frac{(n+1)!}{r!\,(n+1-r)!} - \frac{r \times n!}{r!\,(n+1-r)!}\\ &= \frac{(n+1)! - r \times n!}{r!\,(n+1-r)!}\\ &= \frac{n!\,(n+1-r)}{r!\,(n+1-r)(n-r)!}\\ &= \frac{n!}{r!\,(n-r)!}\\ &= {}^{n}C_r\\ &= LHS\end{aligned}$$

Specific behaviours

✓ correctly obtains factorial expression for RHS
✓ correctly obtains single fraction
✓ completes proof

Question 19 **(8 marks)**

(a) Points A, B, C and D lie on an arc of a circle as shown, so that AD is a diameter.



Let $\alpha = \angle CAD$ and $\beta = \angle ABC$.

Determine in simplest form the relationship between α and β. (3 marks)

Solution
Angles in semicircle: $\angle CAD + \angle ADC = 90° \rightarrow \angle ADC = 90° - \alpha$ Opposite angles in cyclic quadrilateral: $\angle ABC + \angle ADC = 180° \rightarrow \angle ADC = 180° - \beta$ Hence $90° - \alpha = 180° - \beta \rightarrow \beta - \alpha = 90°$.
Specific behaviours
✓ uses angles in semicircle property   ✓ uses opposite angles property   ✓ simplified relationship

$\beta = 90 + \alpha$

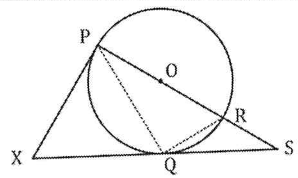
(b) Tangents from X touch a circle at P and Q. Diameter PR and tangent XQ are both extended to meet at S. Prove that $\angle PXQ = 2\angle RQS$. (5 marks)

Solution

1. Angles in semicircle:
$\angle PQR = 90°$
2. Straight angle:
$\angle XQP = 180° - 90° - \angle RQS$
$= 90° - \angle RQS$

3. Tangents from point form isosceles triangle:
$\angle PXQ = 180° - 2\angle XQP$
$= 180° - 2(90° - \angle RQS)$
$= 2\angle RQS$

Hence $\angle PXQ = 2\angle RQS$.



1. Angles in alternate segments:
$\angle QPR = \angle RQS$
2. Tangent-radius is right angle:
$\angle XPQ = 90° - \angle QPR = 90° - \angle RQS$

Specific behaviours

✓ diagram
✓ shows (1) - that $\angle PQR = 90°$ or $\angle QPR = \angle RQS$
✓ shows (2) - that $\angle XQP$ or $\angle XPQ = 90° - \angle RQS$
✓ shows (3) - that $\angle PXQ = 2\angle RQS$
✓ clear reasoning throughout

Supplementary page

Question number: _________